Inertial Platform Stabilization Criteria for Arbitrary Vehicle Maneuvers

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Stability characteristics are described for four gimbal inertial platforms installed on vehicles to provide a fixed attitude reference system. Platform stabilization criteria are formulated that relate arbitrary vehicle maneuvers to 1) the ratio of available motor torque to gimbal inertia reaction, 2) the dynamics of the inner roll gimbal angle when the outer roll servo loop operates below and at motor torque saturation, and 3) vehicle angular acceleration stability regions. The analysis provides a simple method to determine the required ratio of motor torque to gimbal inertia reaction as a function of vehicle motion and gimbal orientation. Once the limits of vehicle motion are known, the torque to inertia ratio can be readily computed to insure maintaining a fixed inertial attitude reference system.

Nomenclature

$\dot{p}$ = vehicle roll acceleration
$\dot{q}$ = vehicle pitch acceleration
$\dot{r}$ = vehicle yaw acceleration
α = inner-roll gimbal travel to stops
β = inner-roll gimbal (limited freedom) angle [the relative angle between the inner-roll and pitch (middle) gimbals]
$\ddot{\beta}$ = inner-roll gimbal relative angle acceleration
θ = pitch gimbal angle (the relative angle between the pitch and outer-roll gimbals)
$\ddot{\theta}$ = pitch gimbal relative angle acceleration
ϕ = outer-roll angle [the relative angle between the outer-roll and case (vehicle) gimbals]
$\ddot{\phi}$ = outer-roll gimbal relative angle acceleration
ψ = azimuth gimbal angle [the relative angle between the azimuth (innermost) and inner-roll gimbals]
$\ddot{\psi}$ = azimuth gimbal relative angle acceleration
${}_2J_z$ = inertia of the inner-roll gimbal about its z axis
${}_3J_x$ = inertia of the pitch gimbal about its x axis
${}_3J_y$ = inertia of the pitch gimbal about its y axis
${}_3J_z$ = inertia of the pitch gimbal about its z axis
${}_4J_x$ = inertia of the outer-roll gimbal about its x axis
$\dot{q}_m$ = maximum vehicle pitch acceleration
$\dot{r}_m$ = maximum vehicle yaw acceleration
$(\dot{q}^2 + \dot{r}^2)_m{}^{1/2}$ = radius of stability
t_0 = instant of time β departs from null
T_β = required inner-roll gimbal motor torque
T_θ = required pitch gimbal motor torque
T_ϕ = required outer-roll gimbal motor torque
T_ψ = required azimuth gimbal motor torque
$(T_\phi/J_\phi)_m$ = maximum available outer-roll torque to inertia ratio
θ_0 = maximum vehicle pitch attitude

Introduction

INERTIAL platforms are dynamically stable and maintain a fixed spatial coordinate system when the azimuth (innermost) gimbal angular velocity components are zero. Four gimbal inertial platforms are instrumented to stabilize the platform by torquing the various gimbals to keep the inner-roll gimbal at null. Ideally, a frictionless, perfectly balanced, drift free, etc., stabilized platform requires no gimbal control torques if the angular momentum of each gimbal, with respect to inertial space, is constant. This condition occurs when the vehicle linear motion, the vehicle attitude reference angles (roll, pitch, and yaw), and the gimbal mass remain constant. Arbitrary vehicle maneuvers produce vehicle angular rates and accelerations which cause the angular momentum of the gimbal system to vary. To preserve the inner gimbal reference, it is necessary for the two outer gimbals to assume a new spatial orientation. The control torques that drive these gimbals to maintain platform stability are a function of vehicle attitude, angular velocity, and acceleration. Platform stability is maintained provided that 1) the outer-roll motor torque is sufficiently large to prevent the inner-roll gimbal from reaching its physical stops, and 2) the pitch motor torque is sufficiently large to prevent vehicle and outer-roll gimbal motion from being transmitted to the stable element if the relative inner-roll gimbal angle β deviates from null. The two inner gimbal torque requirements are independent of vehicle motion and are always zero, neglecting friction, mass unbalance, etc.

Instrumentation of the outer-roll servo loop is such that a torque T_ϕ is produced whenever the relative inner-roll gimbal angle deviates from null. This torque drives the outer-roll gimbal to reduce β to zero. The pitch servo loop is mechanized so that a torque T_θ is produced whenever β deviates from null to drive the pitch gimbal to maintain the inner gimbal reference. The two inner servo loops generate corrective torques primarily to overcome various forces, e.g., friction.

Platform tumbling occurs when the required torque exceeds the maximum available motor torque output, so that the servo loop operates at torque saturation and is unable to prevent the inner-roll gimbal from reaching its physical stops. When the stops are reached, any motion about the free axis of the inner-roll gimbal results in motion of the azimuth gimbal, and thus the inertial reference system is lost.

The mathematical approach presented herein investigates the dynamics of stable four-gimbal inertial platforms. Specifically, the analysis describes the behavior of the inner-roll gimbal relative angle to arbitrary vehicle maneuvers when the outer-roll servo torque motor operates either below or at torque saturation. As a first approximation to the general platform stability problem, idealized platform servo performance is assumed as follows: 1) negligible error source gyro gimbal torques, 2) perfect gyro threshold and rate input sensor capability, and 3) suitable servo loop gimbal, torquer, and pickoff dynamics. For any specific mechanization, the effects and sensitivities of these contributing factors, as well as certain nonlinearities which occur in practice, can be investigated conveniently by digital computer simulation studies.

The theoretical results obtained can greatly assist a development engineer in the design of the electromechanical

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system required for outer-roll follow-up platform stabilization. The vehicle angular acceleration stability criteria derived for the complex dynamics of idealized four-gimbal inertial platforms also will aid digital computer studies to optimize the platform control loops for high-performance vehicles.

Platform Dynamics

The dynamics of an idealized four-gimbal platform tied rigidly to a vehicle are a function of the arbitrary vehicle motion and the control torques applied about the free gimbal axes. Closed-loop platform operation maintains an inertial reference system by introducing corrective torques to drive the outer-roll and pitch gimbals to keep the inner gimbal inertially fixed and the inner-roll gimbal angle at null. The control law, neglecting friction, mass unbalance, motor damping, etc.,[1] is

$$T_\psi = 0 \qquad T_\beta = 0 \qquad T_\theta = ({}_3J_y + {}_2J_z)\ddot{\psi}\sin\beta$$
$$T_\phi = ({}_4J_x + {}_3J_x)\ddot{\beta}\cos\theta - ({}_4J_x + {}_3J_z + {}_2J_z)\ddot{\psi}\cos\beta\sin\theta \tag{1}$$

where

$$\ddot{\psi} = -\frac{(\dot{p} - \ddot{\phi})\sin\theta + (\dot{q}\sin\phi + \dot{r}\cos\phi)\cos\theta}{\cos\beta}$$
$$\ddot{\beta} = (\dot{p} - \ddot{\phi})\cos\theta + (\dot{q}\sin\phi + \dot{r}\cos\phi)\sin\theta$$
$$\ddot{\theta} = [(\dot{p} - \ddot{\theta})\sin\theta - (\dot{q}\sin\phi + \dot{r}\cos\phi)\cos\theta]\tan\beta + (\dot{q}\cos\phi - \dot{r}\sin\phi) \tag{2}$$

From system equations (1) and (2)

$$T_\psi = 0 \qquad T_\beta = 0$$
$$T_\theta = J_\theta[-(\dot{p} - \ddot{\phi})\sin\theta + (\dot{q}\sin\phi + \dot{r}\cos\phi)\cos\theta]\tan\beta$$
$$T_\phi = J_\phi'[(\dot{p} - \ddot{\phi})\cos\theta + (\dot{q}\sin\phi + \dot{r}\cos\phi)\sin\theta]\cos\theta - J_\phi[-(\dot{p} - \ddot{\phi})\sin\theta + (\dot{q}\sin\phi + \dot{r}\cos\phi)\cos\theta]\sin\theta \tag{3}$$

where

$$J_\theta = {}_3J_y + {}_2J_z \qquad J_\phi' = {}_4J_x + {}_3J_x$$
$$J_\phi = {}_4J_x + {}_3J_z + {}_2J_z$$

A nonrotating platform is stabilized, identically aligned with the desired reference system when the inner gimbal angular velocity ${}_1\omega$ is zero. The condition for this to occur is given by

$${}_1\omega = {}_1\omega_x + {}_1\omega_y + {}_1\omega_z = 0 \tag{4}$$

The foregoing is satisfied for four-gimbal platforms when either

$$\beta \equiv 0 \tag{5}$$

or

$$|\beta| < \alpha \tag{6}$$

The condition (5) states that the platform is stable when the inner-roll gimbal angle β is always at null, while (6) asserts the stability criterion when β lies between its physical stops. Rewriting T_ϕ from Eq. (3) results in

$$T_\phi = [J_\phi' - J_\phi][(\dot{p} - \ddot{\phi})\cos^2\theta + (\dot{q}\sin\phi + \dot{r}\cos\phi)\sin\theta\cos\theta] + J_\phi(\dot{p} - \ddot{\phi}) = H + J_\phi(\dot{p} - \ddot{\phi})$$

Solving for $\ddot{\phi}$,

$$\ddot{\phi} = \dot{p} + [(H - T_\phi)/J_\phi] \tag{7}$$

The required outer-roll torque T_ϕ to maintain $\beta \equiv 0$ is derived as follows. Substituting (7) into (2) to obtain

$$\ddot{\beta} = (\dot{q}\sin\phi + \dot{r}\cos\phi)\sin\theta + [(T_\phi - H)/J_\phi]\cos\theta \tag{8}$$

or

$$\ddot{\beta} = (\dot{q}^2 + \dot{r}^2)^{1/2}\sin[\phi + \tan^{-1}(\dot{r}/\dot{q})]\sin\theta + [(T_\phi - H)/J_\phi]\cos\theta \tag{9}$$

and, since $\beta \equiv 0$, $\ddot{\beta} \equiv 0$, and hence (9) becomes

$$T_\phi = -J_\phi(\dot{q}^2 + \dot{r}^2)^{1/2}\sin[\phi + \tan^{-1}(\dot{r}/\dot{q})]\tan\theta + H \tag{10}$$

Similarly, from (3) and (7) the required pitch torque is

$$T_\theta = J_\theta\{-(T_\phi - H)/J_\phi\sin\theta + (\dot{q}^2 + \dot{r}^2)^{1/2}\sin[\phi + \tan^{-1}(\dot{r}/\dot{q})]\cos\theta\}\tan\beta \tag{11}$$

from (10) and (11),

$$T_\theta = J_\theta(\dot{q}^2 + \dot{r}^2)^{1/2}\sin[\phi + \tan^{-1}(\dot{r}/\dot{q})](\tan\beta/\cos\theta) \tag{12}$$

Note that $T_\theta \equiv 0$, when $\beta \equiv 0$. Maximum bounds of the outer-roll and pitch motor torques are obtained by noting that

$$|\sin[\phi + \tan^{-1}(\dot{r}/\dot{q})]| \leq 1 \tag{13}$$

$\therefore$

$$|T_\phi| \leq |J_\phi(\dot{q}^2 + \dot{r}^2)^{1/2}\tan\theta| + |H| \tag{14}$$
$$|T_\theta| \leq |J_\theta(\dot{q}^2 + \dot{r}^2)^{1/2}(\tan\beta/\cos\theta)| \tag{15}$$

A bound on the function H, when β is identically zero, is obtained as follows. From Ref. 1,

$$\dot{p} - \ddot{\phi} = -\dot{\psi}\dot{\theta}\cos\theta - \ddot{\psi}\sin\theta \tag{16}$$
$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta \tag{17}$$
$$\dot{\theta} = q\cos\phi - r\sin\phi \tag{18}$$

From Eqs. (16–18, and 2),

$$(\dot{p} - \ddot{\phi}) = \frac{(q\sin\phi + r\cos\phi)(r\sin\phi - q\cos\phi) - (\dot{q}\sin\phi + \dot{r}\cos\phi)\cos\theta\sin\theta}{\cos^2\theta}$$

Substituting $(\dot{p} - \ddot{\phi})$ from the aforementioned into H yields

$$H = [J_\phi' - J_\phi][(q\sin\phi + r\cos\phi)(r\sin\phi - q\cos\phi)]$$
$$= [J_\phi' - J_\phi]\{[(r^2 - q^2)\sin 2\phi - 2rq\cos 2\phi]/2\}$$
$$|H| \leq |J_\phi' - J_\phi|\left[\frac{r^2 + q^2 + 2|rq|}{2}\right]$$
$$|H| \leq \frac{|J_\phi' - J_\phi|[|r| + |q|]^2}{2} \tag{19}$$

Stability Criteria (β at Null)

Platform stability ($\beta \equiv 0$) is maintained provided that

$$\left(\frac{T_\phi}{J_\phi}\right) \geq (\dot{q}^2 + \dot{r}^2)^{1/2}|\tan\theta| + \left|\frac{H}{J_\phi}\right| \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \tag{20}$$

and

$$T_\phi/J_\theta = 0 \tag{21}$$

where $(T_\phi/J_\phi)_m$ = the maximum available outer-roll torque to inertia ratio. The maximum vehicle pitch attitude, at which platform stability is maintained when the maximum vehicle pitch $\dot{q}_m$ and yaw $\dot{r}_m$ accelerations are in effect, is $\theta_m(t_0)$ where

$$\theta_m(t) = \tan^{-1}\left[\frac{(T_\phi/J_\phi)_m - |H(t)/J_\phi|}{(\dot{q}_m{}^2 + \dot{r}_m{}^2)^{1/2}}\right]$$

and t_0 is such that

$$\theta_m(t) > \theta_m(t_0) \quad \text{when} \quad \beta \neq 0 \quad t_0 < t < t_0 + \epsilon$$

$$\theta_m(t) \leq \theta_m(t_0) \quad \text{when} \quad \beta \equiv 0 \quad t \leq t_0$$

$\therefore$

$$\theta_m(t_0) \leq \tan^{-1}\left[\frac{(T_\phi/J_m)_m}{(\dot{q}_m^2 + \dot{r}_m^2)^{1/2}}\right] \tag{22}$$

The maximum circular radius of stability $(\dot{q}^2 + \dot{r}^2)_m{}^{1/2}$ defines the circular region of the vehicle pitch and yaw accelerations within which platform stability is maintained for a vehicle pitch attitude θ_0 at most

$$\theta_0 \leq \tan^{-1}\left[\frac{(T_\phi/J_\phi)_m}{(\dot{q}^2 + \dot{r}^2)m^{1/2}}\right] \tag{23}$$

Note that $\theta_0 = \theta_m$, when $(\dot{q}^2 + \dot{r}^2)_m{}^{1/2} = (\dot{q}_m{}^2 + \dot{r}_m{}^2)^{1/2}$ and if $\theta<\theta_0$ then $\beta \equiv 0$ and the outer-roll servo loop operates below torque saturation, while if $\theta>\theta_0$ then β departs from its null and the outer-roll servo loop operates at torque saturation.

Stability Criteria (β between Its Physical Stops)

A vehicle maneuver that causes β to reach its physical stops will result in loss of the inertial reference system. To determine a maximum bound on β from the time T that it deviates from null, the following analytical technique is employed:

$$d[\dot{\beta}^2] = 2\ddot{\beta}(d\beta/d\theta)d\theta \tag{24}$$

An upper bound for $\ddot{\beta}$ from (9) is

$$\ddot{\beta} \leq (\dot{q}^2 + \dot{r}^2)^{1/2}\sin\theta + \left|\frac{(T_\phi - H)}{J_\phi}\right|\cos\theta \qquad \left(0 \leq \theta \leq \frac{\pi}{2}\right) \tag{25}$$

Since $\theta = f(t)$ strictly increases with time, its inverse, $t = g(\theta)$, exists. Replacing t by $g(\theta)$ in (25)

$$\ddot{\beta} \leq (\dot{q}^2 + \dot{r}^2)^{1/2}\sin\theta + |(T_\phi - H)/J_\phi|\cos\theta = F(\theta) \tag{26}$$

An alternate expression for $F(\theta)$ is given by

$$F(\theta) = \{(\dot{q}^2 + \dot{r}^2) + [(T_\phi - H)/J_\phi]^2\}^{1/2}\sin(\theta + \theta_k) \tag{27}$$

where

$$\theta_k = \tan^{-1}\left[\frac{|[(T_\phi - H)/J_\phi]|}{(\dot{q}^2 + \dot{r}^2)^{1/2}}\right]$$

At the instant that β departs from null, $\theta_k = \theta_0$ and

$$(\dot{q}^2 + \dot{r}^2)^{1/2} = (\dot{q}^2 + \dot{r}^2)_m$$

Thus Eq. (24) is written as

$$\int_{\theta_0}^{\theta} d(\dot{\beta}^2) \leq 2\int_{\theta_0}^{\theta} F(\theta)\frac{d\beta}{d\theta}d\theta \tag{28}$$

Note that in the interval of integration $[\theta_0,\theta]$ the outer-roll servo loop operates at torque saturation and β deviates from null. By the first law of the mean of the Integral Calculus,

$$2\int_{\theta_0}^{\theta} F(\theta)\frac{d\beta}{d\theta}d\theta = 2F(\hat{\theta})\int_{\theta_0}^{\theta}\frac{d\beta}{d\theta}d\theta$$

provided that

$$\theta_0 \leq \theta \leq \theta_L = \pi/2 \qquad \hat{\theta} = \hat{\theta}(\theta) = \hat{\theta}_1(t)$$

$d\beta/d\theta$ does not change sign and is integrable in (θ_0,θ), while $F(\theta)$ is continuous. From Eq. (27) $F(\theta)$ is continuous. If the function $(d\beta/d\theta)$ oscillates between its physical stops, the platform is stable. For any interval in which β is monotonic, $d\beta/d\theta$ is of one sign, and hence applying the first law of the mean to (28),

$$\dot{\beta}^2(\theta) - \dot{\beta}^2(\theta_0) \leq 2F(\hat{\theta})[\beta(\theta) - \beta(\theta_0)]$$

since

$$\begin{aligned} \beta(\theta_0) &= \dot{\beta}(\theta_0) = 0 \\ \dot{\beta}^2(\theta) &\leq 2F(\hat{\theta})\beta(\theta) \qquad [\beta(\theta) \geq 0] \\ \dot{\beta}(\theta)/\beta^{1/2}(\theta) &\leq [2F(\hat{\theta})]^{1/2} \\ \int_{t_0}^{t}\frac{\dot{\beta}(t)}{\beta^{1/2}(t)}dt &\leq \int_{t_0}^{t}[2F(\hat{\theta})]^{1/2}dt \end{aligned} \tag{29}$$

where t_0 = the instant of time that β departs from null, and t = the time for β to reach its stops from t_0. From Eq. (27), the maximum value of $F(\theta)$ in the interval of interest $[\theta_0, 90°]$ is

$$F_{\max} = \{(\dot{q}^2 + \dot{r}^2)_m + [(T_\phi/J_\phi)_m - (H/J_\phi)]^2\}^{1/2} \times \sin[(\pi/2) - \theta_0] \geq F(\hat{\theta}) \tag{30}$$

Integrating both sides of Eq. (29) yields

$$\begin{aligned} 2\beta^{1/2}(t) - 2\beta^{1/2}(t_0) &\leq \int_{t_0}^{t}[2F(\hat{\theta})]^{1/2}dt \leq \int_{t_0}^{t}(2F_{\max})^{1/2}dt \\ 2\beta^{1/2}(t) &\leq (2F_{\max})^{1/2}(t - t_0) = T(2F_{\max})^{1/2} \\ \beta(t) &\leq F_{\max}T^2/2 \end{aligned} \tag{31}$$

since

$$\sin\left(\frac{\pi}{2} - \theta_0\right) = \cos\theta_0 = \frac{(\dot{q}^2 + \dot{r}^2)_m{}^{1/2}}{\{(\dot{q}^2 + \dot{r}^2)_m + [(T_\phi/J_\phi)_m - H/J_\phi]^2\}^{1/2}} \tag{32}$$

From Eqs. (30–32),

$$\beta(t) \leq (\dot{q}^2 + \dot{r}^2)_m{}^{1/2}T^2/2 \tag{33}$$

$$T \geq (2)^{1/2}\beta^{1/2}(t)/(\dot{q}^2 + \dot{r}^2)_m{}^{1/2} \tag{34}$$

The expression for β [Eq. (33)] yields the maximum departure from null during the time T that the servo loop operates at torque saturation. Inversely, the minimum time for the inner-roll gimbal to depart β rad is given by Eq. (34). Both parameters are valid for all pitch attitude angles (θ_0). At the zenith infinite outer-roll torque is required to maintain β at null. With finite torques, as θ_0 passes through the zenith, β will deviate from null as indicated by Eq. (33). Stability is retained provided that the inner-roll gimbal stops are not reached.

Reference

[1] Macomber, G. R. and Fernandez, R., *Inertial Guidance Engineering* (Prentice-Hall Inc., Englewood Cliffs, N. J., 1962), Chap. II, pp. 101–115.